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High-density quark matter under stress

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Abstract

We study the effect of $SU(3)$ -flavor breaking on high-density quark matter. We discuss, in particular, a nonzero electron chemical potential and a finite strange-quark mass. We argue that these perturbations trigger pion or kaon condensation. The critical chemical potential behaves as $\mu_e \sim \sqrt{m_s} \Delta / p_F$ and the critical strange-quark mass as $m_s \sim m^{1/3} \Delta^{2/3}$, where m is the light-quark mass, Δ is the gap, and p_F is the Fermi momentum. We note that parametrically, both the critical μ_e and $m_s^2 / (2p_F)$ are much smaller than the gap. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

The study of hadronic matter in the regime of high baryon density and small temperature has revealed a rich and beautiful phase structure [1–3]. One phase which has attracted particular interest is the color–flavor locked (CFL) phase of three-flavor quark matter [4]. This phase is expected to be the true ground state of ordinary matter at very high density [5–7]. State of the art calculations are not sufficiently accurate to predict the critical density of the transition to CFL matter with any certainty. Current estimates typically range as $\rho_{\text{crit}} \sim (3–6)\rho_0$, where ρ_0 is the saturation density of nuclear matter. An exciting prospect is the possibility to put experimental constraints on the critical density from observations of neutron stars. Several proposals have been made for observables that are characteristic of different superfluid quark phases, and attempts are being made in order to include these phases in realistic neutron star structure calculations [8–10].

Initial work on the superfluid phases of QCD focused mostly on idealized worlds with N_f flavors of massless fermions and no external fields. But in order to understand the

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matter at the core of real neutron stars the effects of nonzero masses and finite chemical potentials clearly have to be taken into account. The first study of the effects of a nonzero strange-quark mass on CFL quark matter was carried out in [11,12]. The main observation in this work was that a finite strange-quark mass shifts the Fermi momentum of the strange quark with respect to the Fermi momentum of the light quarks. If the mismatch between the Fermi momenta is bigger than the gap then pairing between strange and nonstrange quarks is no longer possible. The transition from CFL matter to quark matter with separate pairing among light and strange quarks ($2 + 1$ SC) is predicted to occur at $m_s \sim \sqrt{p_F \Delta}$. Alford et al. observed that in the vicinity of this phase transition we expect to encounter inhomogeneous BCS phases [13] analogous to the Larkin–Ovchinnikov–Fulde–Ferrell (LOFF) phase in condensed matter physics [14–16]. In the LOFF phase Cooper pairs have nonzero total momentum and, as a consequence, pairing is restricted to certain regions of the Fermi surface.

In the present work we analyze CFL matter for strange-quark masses and chemical potentials below the unlocking transition [17]. We will argue that in this regime CFL matter responds to the external “stress” by forming a Bose condensate of kaons or pions [18]. This effect can be understood as a chiral rotation of the CFL order parameter. Superfluid quark matter composed of only two flavors is characterized by an order parameter $\langle \epsilon^{abc} u^b C \gamma_5 d^c \rangle$ which is a flavor singlet [19–21]. This order parameter is “rigid” and superfluidity has to be destroyed in order to create a macroscopic occupation number of charged excitations [22]. CFL matter, on the other hand, is characterized by an order parameter which is a matrix in color and flavor space [4]:

$$\langle q_{L,i}^a C q_{L,j}^b \rangle = -\langle q_{R,i}^a C q_{R,j}^b \rangle = \phi (\delta_i^a \delta_j^b - \delta_i^b \delta_j^a), \quad (1)$$

where i, j label flavor and a, b label color indices. We can introduce a chiral field Σ that characterizes the relative flavor orientation of the left- and right-handed condensates [23]. In the vacuum $\Sigma = 1$, but under the influence of a perturbation Σ may rotate. Because Σ has the quantum numbers of pseudoscalar Goldstone bosons, such a rotation corresponds to a macroscopic occupation number of Goldstone bosons.

There is an even simpler way to explain the phenomenon of kaon condensation in superfluid quark matter, see Fig. 1. Here we concentrate on the effect of a nonzero strange-quark mass. A nonzero quark-mass shifts the energy of strange quarks in the vicinity of the Fermi surface by $\sim m_s^2/(2p_F)$. In normal quark matter this leads to the decay $s \rightarrow u + e^- + \bar{\nu}_e$ (or $s \rightarrow u + d + \bar{u}$). This decay will reduce the number of strange quarks and build up a Fermi sea of electrons until the electron chemical potential reaches $\sim m_s^2/(4p_F)$. In superfluid quark matter the system can also gain energy $m_s^2/(2p_F)$ by introducing an extra up-quark and a strange hole. This process appears to require the breaking of a pair and therefore involve an energy cost which is of the order of the gap Δ . This is not correct, however. An up-, down-particle/strange-hole pair has the quantum numbers of a kaon. This means that the energy cost is not Δ , but $m_K \ll \Delta$. The CFL vacuum can decay into K^+ or K^0 collective modes via processes like $0 \rightarrow (\bar{d}s)(du) + e^- + \bar{\nu}_e$ or $0 \rightarrow (\bar{u}s)(du)$.

This paper is organized as follows. In Section 2 we present general arguments for the existence of kaon and pion condensates in high-density matter with broken flavor

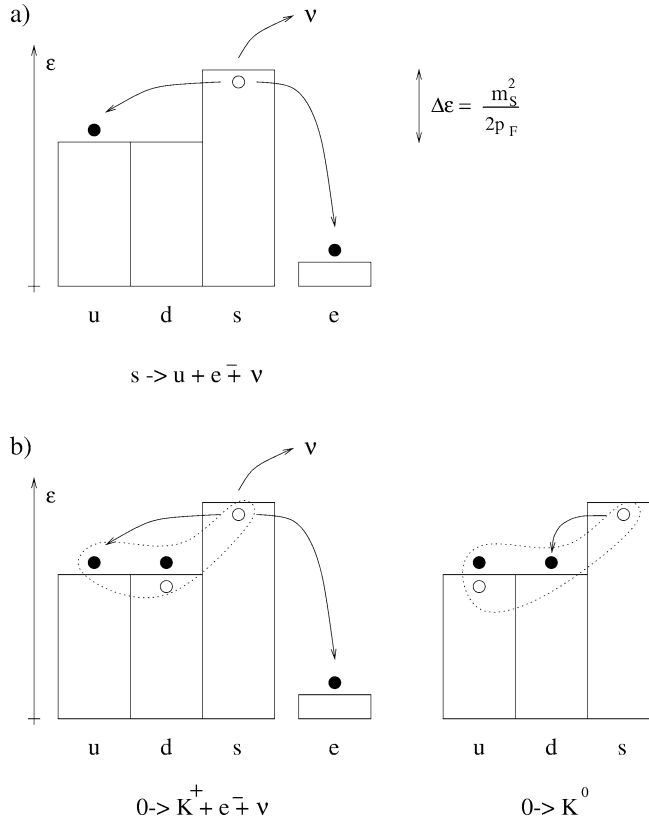


Fig. 1. Schematic picture of weak decays in normal (a) and superfluid (b) quark matter with three quark flavors. We assume that initially the density of all quark flavors is the same, so that $\epsilon_{F,s} \simeq \epsilon_{F,ud} + m_s^2/(2p_F)$. Solid and open circles show particles (p) and holes (h). In (a), a strange particle decays into an up-quark, an electron and a neutrino, leaving behind a strange hole. In the left panel of (b), a strange particle decays into an up-quark, a down-particle-hole pair, an electron and a neutrino. The remaining (pp)(hh) configuration has the quantum numbers of a K^+ . In the right panel we show the decay of a strange quark into a (pp)(hh) configuration with the quantum numbers of a K^0 .

symmetry. In Section 3 we strengthen these arguments by performing an explicit matching calculation. In Section 4 we provide a different perspective on our results by using linear response theory.

2. Three-flavor quark matter at $m_s \neq 0$ and $\mu_e \neq 0$

In order to study QCD at high baryon density it is convenient to use an effective description that focuses on excitations close to the Fermi surface. Two effective descriptions of this type are available. The first effective theory is valid for excitation energies below the Fermi momentum p_F , while the second one applies to excitation energies below the gap Δ . The coefficients that appear in these effective theories can be worked out using matching

arguments. In the first stage we match the microscopic theory, QCD at finite baryon density, to an effective theory below p_F . In the second step, we match this effective description to an effective theory involving Goldstone modes.

The QCD Lagrangian in the presence of a chemical potential is given by

$$\mathcal{L} = \bar{\psi}(\not{D} + \mu\gamma_0 - \mu_e Q\gamma_0)\psi - \bar{\psi}_L M \psi_R - \bar{\psi}_R M^\dagger \psi_L - \frac{1}{4}G_{\mu\nu}^a G_{\mu\nu}^a, \quad (2)$$

where M is a complex quark-mass matrix which transforms as $M \rightarrow L M R^\dagger$ under chiral transformations $(L, R) \in SU(3)_L \times SU(3)_R$, Q is the quark-charge matrix, μ is the baryon chemical potential and μ_e is (minus) the chemical potential for electric charge. As usual, we treat M as a (spurious) field in order to determine the structure of mass terms in the effective chiral theory. Once this has been achieved, we set the mass matrix to its physical value $M = \text{diag}(m_u, m_d, m_s)$.

The quark field ψ can be decomposed as $\psi = \psi_+ + \psi_-$ where $\psi_\pm = \frac{1}{2}(1 \pm \vec{\alpha} \cdot \hat{p})\psi$. The ψ_+ component of the field describes quasiparticle excitations in the vicinity of the Fermi surface. Integrating out the ψ_- field we get [24–26]:

$$\begin{aligned} S = \int \frac{d^4p}{(2\pi)^4} \frac{d^3p}{(2\pi)^3} \left\{ \psi_{L+}^\dagger (p_0 - \epsilon_p - v \cdot A) \psi_{L+} \right. \\ - \frac{\Delta}{2} (\psi_{L+}^{ai} C \psi_{L+}^{bj} (\delta_{ai} \delta_{bj} - \delta_{aj} \delta_{bi}) + \text{h.c.}) \\ + \psi_{L+}^\dagger \left(-\mu_e Q - \frac{M M^\dagger}{2p_F} \right) \psi_{L+} \\ + \frac{\bar{\Delta}}{8p_F^2} \psi_{L+}^{ai} C \psi_{L+}^{bj} (M_{ai}^\dagger M_{bj}^\dagger - M_{aj}^\dagger M_{bi}^\dagger) \\ \left. + (R \rightarrow L, M \rightarrow M^\dagger, Q \rightarrow Q^\dagger) + \dots \right\}, \quad (3) \end{aligned}$$

where $\epsilon_p = |\vec{p}| - \mu$, $v_\mu = (1, \vec{v})$ with $\vec{v} = \vec{p}/p$, $\bar{\Delta}$ is a parameter that controls mass corrections to the gap and i, j, \dots and a, b, \dots denote flavor and color indices. In order to perform perturbative calculations in the superconducting phase we have added a tree-level gap term $\psi_{L,R} C \Delta \psi_{L,R}$ in the free part of the Lagrangian and subtracted it from the interacting part (not explicitly shown). The magnitude of Δ can be determined self consistently order by order in perturbation theory. In the normal phase, both Δ and $\bar{\Delta}$ vanish. In this case, only the first mass term in (3) contributes.

We observe that at $O(1/p_F)$, flavor symmetry breaking due to a chemical potential for charge is indistinguishable from symmetry breaking due the quark-mass matrix. Indeed, up to terms suppressed by additional powers of (Δ/p_F) , (p/p_F) or (m/p_F) the Lagrangian (3) is invariant under the time-dependent flavor symmetry (from now on we drop the subscript “+”):

$$\begin{aligned} \psi_L &\rightarrow L(t) \psi_L, \\ \psi_R &\rightarrow R(t) \psi_R, \\ \left(-\mu_e Q - \frac{M M^\dagger}{2p_F} \right) &\rightarrow L(t) \left(-\mu_e Q - \frac{M M^\dagger}{2p_F} \right) L^\dagger(t) + iL(t) \partial_0 L^\dagger(t), \end{aligned}$$

$$\left(-\mu_e Q^\dagger - \frac{M^\dagger M}{2p_F}\right) \rightarrow R(t) \left(-\mu_e Q^\dagger - \frac{M^\dagger M}{2p_F}\right) R^\dagger(t) + iR(t) \partial_0 R^\dagger(t), \quad (4)$$

where $L(t)$ and $R(t)$ are left- and right-handed time-dependent flavor transformations.

For excitation energies below the gap Δ , we can use an effective theory that includes only the pseudo-Goldstone bosons [23,26–28]. The scale of the momentum and energy expansion in this theory is set by the gap Δ . Taking into account the symmetries discussed above we see that a generic term in the effective Lagrangian has the form

$$\mathcal{L} \sim f_\pi^2 \Delta^2 \left(\frac{\partial_0 - i\mu_e Q - iMM^\dagger/(2p_F)}{\Delta} \right)^n \left(\frac{\vec{\partial}}{\Delta} \right)^m \left(\frac{MM}{p_F^2} \right)^p \left(\frac{\mu_e Q}{p_F} \right)^q. \quad (5)$$

This equation implies that the N th-order term in the effective Lagrangian is given by the most general $SU(3)_L \times SU(3)_R$ invariant term constructed from the chiral field Σ and containing n covariant time derivatives, m spatial derivatives, p powers of M^2 , and q powers of $\mu_e Q$ such that $N = n + m + p + q$. We note that mass terms are suppressed by either M^2/p_F^2 or $MM^\dagger/(p_F \Delta)$. Terms of the form M^2/p_F^2 contain the quark-mass matrix in the flavor antisymmetric combination shown in the gap term in Eq. (3).

The leading terms of the effective Lagrangian take the form

$$\mathcal{L}_{\text{eff}} = \frac{f_\pi^2}{4} \text{Tr}[\nabla_0 \Sigma \nabla_0 \Sigma^\dagger - v_\pi^2 \partial_i \Sigma \partial_i \Sigma^\dagger] + 2A [\det(M) \text{Tr}(M^{-1} \Sigma) + \text{h.c.}] + \dots, \quad (6)$$

$$\nabla_0 \Sigma = \partial_0 \Sigma + i \left(\mu_e Q + \frac{MM^\dagger}{2p_F} \right) \Sigma - i \Sigma \left(\mu_e Q^\dagger + \frac{M^\dagger M}{2p_F} \right). \quad (7)$$

Here $\Sigma = \exp(i\pi^a \lambda^a / f_\pi)$ is the flavor-octet chiral field and the $SU(3)_A$ generators are normalized as $\text{Tr}[\lambda^a \lambda^b] = 2\delta^{ab}$. We have not displayed the flavor-singlet part of the effective Lagrangian. The first term in Eq. (6) is invariant under the approximate symmetry (4) because of the presence of the covariant time derivative. The second term is not invariant under (4), but $A \sim f_\pi^2 \Delta^2 / p_F^2$ is suppressed by $1/p_F^2$, in accordance with Eq. (5).

The M^2 term is not the most general term consistent with the symmetries. The structure of this term determined by the fact that it has to contain the quark-mass matrix in a flavor antisymmetric combination. $O(M^2)$ terms that are symmetric in flavor do not vanish, but they are strongly suppressed. We provide an estimate of these terms in Appendix A.

Despite the similarity between the effective theory for the Goldstone modes in the CFL phase and chiral perturbation theory in vacuum, there are important differences in the power counting. As usual, the contribution of loops is suppressed by powers of p/f_π . However, in the CFL phase, $f_\pi \sim p_F \gg \Delta$ which means that the suppression of loops with respect to tree level terms is much more pronounced than it is in the vacuum.

More differences appear in the expansion in M . First of all, because of an approximate axial Z_2 symmetry in the CFL phase there are no odd powers in M . In addition to that, the $(MM^\dagger)(M^\dagger M)$ terms can become comparable to the M^2 terms without breaking the chiral expansion. Indeed, as we shall argue below, this is likely to be the case for realistic values of m_s and p_F . There are two reasons why the $(MM^\dagger)(M^\dagger M)$ term can become comparable to the M^2 term. First, the term proportional to $(MM^\dagger)(M^\dagger M)$ gives a contribution to

meson masses which is of the order m^2/p_F while the M^2 term contributes at order $m\Delta/p_F$. These contributions are comparable if $m \sim \Delta$, which is inside the regime of validity of the effective theory, $m < \sqrt{\Delta p_F}$. Second, in the realistic case where $m_s \gg m_d, m_u$, the term quadratic in M is proportional to at least one light-quark mass, while the term quartic in M contains terms proportional to m_s^4 .

Using (6) we can easily compute the masses of the Goldstone bosons in the CFL phase. At large density, Lorentz invariance is broken and we identify the mass with the energy of a $\vec{p} = 0$ mode. For $\mu_e = 0$ the masses of the flavor nonsinglet states are given by

$$\begin{aligned} m_{\pi^\pm} &= \mp \frac{m_d^2 - m_u^2}{2p_F} + \left[\frac{4A}{f_\pi^2} (m_u + m_d) m_s \right]^{1/2}, \\ m_{K^\pm} &= \mp \frac{m_s^2 - m_u^2}{2p_F} + \left[\frac{4A}{f_\pi^2} m_d (m_u + m_s) \right]^{1/2}, \\ m_{K^0, \bar{K}^0} &= \mp \frac{m_s^2 - m_d^2}{2p_F} + \left[\frac{4A}{f_\pi^2} m_u (m_d + m_s) \right]^{1/2}. \end{aligned} \quad (8)$$

The splitting between particles and antiparticles can be understood by observing that the crossed terms in the kinetic term of Eq. (3) act as an effective chemical potential for strangeness/isospin even if $\mu_e = 0$. We observe that the pion masses are not strongly affected but the mass of the K^+ and K^0 is substantially lowered while the K^- and \bar{K}^0 are pushed up. As a result the K^+ and K^0 meson become massless if $m_s \sim m_{u,d}^{1/3} \Delta^{2/3}$. For larger values of m_s the kaon modes are unstable, signaling the formation of a kaon condensate.

Once kaon condensation occurs the ground state is reorganized. For simplicity, we consider the case of exact isospin symmetry $m_u = m_d \equiv m$. The most general ansatz for a kaon condensed ground state is given by

$$\begin{aligned} \Sigma = \exp(i\alpha [&\cos(\theta_1)\lambda_4 + \sin(\theta_1)\cos(\theta_2)\lambda_5 + \sin(\theta_1)\sin(\theta_2)\cos(\phi)\lambda_6 \\ &+ \sin(\theta_1)\sin(\theta_2)\sin(\phi)\lambda_7]). \end{aligned} \quad (9)$$

With this ansatz the vacuum energy is given by

$$V(\alpha) = -f_\pi^2 \left(\frac{1}{2} \left(\frac{m_s^2 - m^2}{2p_F} \right)^2 \sin(\alpha)^2 + (m_K^0)^2 (\cos(\alpha) - 1) \right), \quad (10)$$

where $(m_K^0)^2 = (4A/f_\pi^2) m_{u,d} (m_{u,d} + m_s)$ is the $O(M^2)$ kaon mass in the limit of exact isospin symmetry. Minimizing the vacuum energy we obtain $\alpha = 0$ if $m_s^2/(2p_F) < m_K^0$ and $\cos(\alpha) = (m_K^0)^2/\mu_{\text{eff}}^2$ with $\mu_{\text{eff}} = m_s^2/(2p_F)$ if $\mu_{\text{eff}} > m_K^0$. We observe that the vacuum energy is independent of θ_1, θ_2, ϕ even if $\alpha \neq 0$. This implies that the effective potential in the kaon condensed phase has three flat directions. The hypercharge density is given by

$$n_Y = f_\pi^2 \mu_{\text{eff}} \left(1 - \frac{m_K^4}{\mu_{\text{eff}}^4} \right), \quad (11)$$

where $\mu_{\text{eff}} = m_s^2/(2p_F)$. This result is typical of a weakly coupled Bose gas [29–31]. We also note that within the range of validity of the effective theory, $\mu_{\text{eff}} < \Delta$, the hypercharge

density satisfies $n_Y < \Delta p_F^2 / (2\pi)$. The upper bound on the hypercharge density in the condensate is equal to the particle density contained within a strip of width Δ around the Fermi surface.

The symmetry breaking pattern is $SU(2)_I \times U(1)_Y \rightarrow U(1)$ where I is isospin and Y is hypercharge. It is amusing to note that this is the symmetry-breaking pattern of the standard model. Kaon condensation is analogous to electroweak symmetry breaking with a composite Higgs field [32,33]. We can discuss kaon condensation in terms of an effective field theory which only involves a complex kaon doublet $\Phi = (K^+, K^0)$

$$\mathcal{L} = [(\partial_0 + i\mu_{\text{eff}})\Phi^\dagger][(\partial_0 - i\mu_{\text{eff}})\Phi] - (m_K^0)^2(\Phi^\dagger\Phi) - \lambda(\Phi^\dagger\Phi)^2. \quad (12)$$

If $\mu_{\text{eff}} > m_K^0$ the kaon field acquires a nonzero vacuum expectation value $\langle\Phi\rangle = (0, v)$ and the $SU(2) \times U(1)$ symmetry is broken to $U(1)$. From (12) we get $v = (\mu_{\text{eff}}^2 - (m_K^0)^2)/(2\lambda)$. We can fix λ by comparing the amplitude of the kaon field to the result obtained from the chiral theory. We find $\lambda = (m_K^0)^2/(2f_\pi^2)$.

In weak coupling the coefficients of the effective Lagrangian can be computed and more quantitative statements about the onset of kaon condensation can be made. The gap is given by [6,34–37]

$$\Delta = 512\pi^4 2^{-1/3} (2/3)^{-5/2} \mu g^{-5} \exp\left(\frac{3\pi^2}{\sqrt{2}g}\right). \quad (13)$$

The pion-decay constant f_π has been computed to leading order in α_s [27] (a factor 2 discrepancy in the literature will be resolved in Section 3):

$$f_\pi^2 = \frac{21 - 8 \log 2}{18} \frac{\mu^2}{2\pi^2}. \quad (14)$$

There is also disagreement about the value of the constant A [26–28,38,39]. The results given in [26,27] are respectively

$$A = \frac{\Delta \bar{\Delta}}{4\pi^2} \log(\mu/\Delta), \quad A = \frac{3\Delta^2}{4\pi^2}. \quad (15)$$

Using the first of these two results, a K^0 condensate forms if

$$m_s^3 > \left(\frac{144}{21 - 8 \log 2}\right) m_u \Delta \bar{\Delta} \log(\mu/\Delta). \quad (16)$$

In Fig. 2 we show the dependence of the kaon mass on m_s for $p_F = 500$ MeV and with Δ and f_π calculated to leading order in perturbation theory. We observe that the K^0 becomes massless for $m_s \simeq 60$ MeV. There is obviously some uncertainty associated with the use of first-order perturbation theory. An estimate of this uncertainty is provided by the scale dependence of the result. We have calculated m_K with g evaluated at the scale $\Lambda = p_F$. Varying Λ between $p_F/2$ and $2p_F$ gives critical strange-quark masses between 39 and 67 MeV.

If charge neutrality is enforced, we have to add the contribution of electrons to the thermodynamic potential, $\Omega(\Sigma, \mu_e) = \Omega_{\text{GB}}(\Sigma, \mu_e) - \mu_e^4/(12\pi^2)$. The ground state is determined by minimizing Ω with respect to Σ subject to the condition that

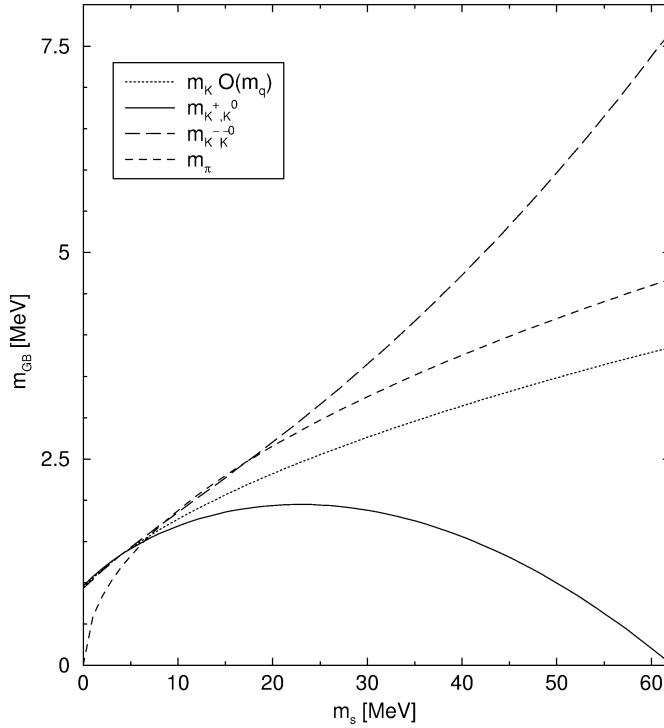


Fig. 2. Masses of K^\pm and K^0, \bar{K}^0 excitations in the color-flavor locked phase. We show the excitation energies as a function of m_s for $p_F = 500$ MeV. The gap $\Delta = 67$ MeV and the pion-decay constant $f_\pi = 104$ MeV were determined to leading order in perturbation theory. The solid and dashed curve show the masses of the (K^+, K^0) and (K^-, \bar{K}^0) states. The dotted curve shows the kaon masses calculated from the leading-order $O(m_q)$ term. The short dashed curve shows the pion masses.

$\partial\Omega/(\partial\mu_e) = 0$. In the isospin symmetric limit these conditions are satisfied by pure K^0 condensation with α as determined above and $\sin(\theta_1) = \sin(\theta_2) = 1$. This conclusion remains valid in the case $m_d > m_u$ because the light-quark mass difference also disfavors K^+ condensation compared to K^0 condensation.

The effect of a small electron chemical potential can also be read off from Eq. (6). A positive electron chemical potential lowers the energy of negatively charged Goldstone modes and increases the energy of positively charged modes:

$$E_{\pi^\pm} = \pm\mu_e + m_{\pi^\pm}, \quad E_{K^\pm} = \pm\mu_e + m_{K^\pm}. \quad (17)$$

A meson condensate will form when μ_e equals the mass of the lightest negatively charged state. Let us again consider the limit of exact isospin symmetry, $m_u = m_d = m$. The mass of the K^- is $m_{K^-} = (2\sqrt{A}/f_\pi)\sqrt{mm_s} + m_s^2/(2p_F)$ and the mass of the π^- is $m_{\pi^-} = \sqrt{2}(2\sqrt{A}/f_\pi)\sqrt{mm_s}$. For very small m_s , the lightest negatively charged particle is the K^- , but for $m_s^2/(2p_F) > (\sqrt{2} - 1)(2\sqrt{A}/f_\pi)\sqrt{mm_s}$ the lightest negative state is the π^- . For negative electron chemical potentials, a K^+ condensate is always favored. We should note that the masses of charged Goldstone bosons are modified by electromagnetic effects. The

electromagnetic self-energy in the CFL phase was estimated to be $m_{\text{em}}^2 \sim \alpha_{\text{em}} \Delta^2$ [40,41]. At sufficiently large baryon density this effect will dominate over the $O(M^2)$ contribution to the Goldstone boson masses.

3. Matching calculation for the $O(M^4)$ terms

In the weak coupling regime the coefficients appearing in the Lagrangian (6) can be computed by matching to perturbative QCD. In this section we will perform the matching calculation for the M^4 terms in Eq. (6). Our goal is two-fold: to strengthen and illustrate the symmetry arguments presented in the previous section and to clarify the calculations of f_π in the literature.¹

We begin by calculating the one-loop polarization functions for the zeroth component of left-handed flavor currents j_L , right-handed flavor currents j_R and (transposed) color currents j_c^T . In the limit $\omega = 0$, $k \rightarrow 0$, we find

$$\Pi_{00}^{AB}(0) = - \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} m_D^2, \quad (18)$$

where the indices A, B correspond to (j_L, j_R, j_c^T) and we have introduced the quantity

$$m_D^2 = \frac{21 - 8 \log(2)}{18} \left(\frac{\mu^2}{2\pi^2} \right), \quad (19)$$

which is, up to a factor g^2 , the Debye mass [27,42]. The LL and RR components of (18) receive contributions both from diagrams with normal propagators and from diagrams with anomalous propagators, see Fig. 3. The LC and RC components only receive contributions from diagrams with anomalous propagators [41]. The overall coefficient is nevertheless exactly the same. The CC entry is twice bigger than the LL and RR entries because it receives contributions from both left- and right-handed fermions.

The matrix (18) is not diagonal, so there is mixing between gluons and left- or right-handed flavor currents. Also, there is no mixing between left- and right-handed flavor currents, contrary to what we would expect for a system with broken chiral symmetry. These defects can be cured by resumming bubble chains with intermediate gluons. In practice we only have to compute the two-loop contribution because higher-order diagrams simply correspond to replacing the free gluon propagator $1/(\omega^2 - k^2)$ with the dressed propagator $1/(\omega^2 - k^2 - g^2 m_D^2)$. The two-loop contributions to the polarization function are superficially suppressed by a factor g^2 , but in the limit $\omega, k \rightarrow 0$ the factor g^2 in the numerator is canceled by the screening mass $g^2 m_D^2$ in the denominator.

Summing all bubble chains, we get

$$\Pi_{00}^{AB}(0) = - \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} m_D^2. \quad (20)$$

¹ We thank D. Kaplan for suggesting this calculation to us.

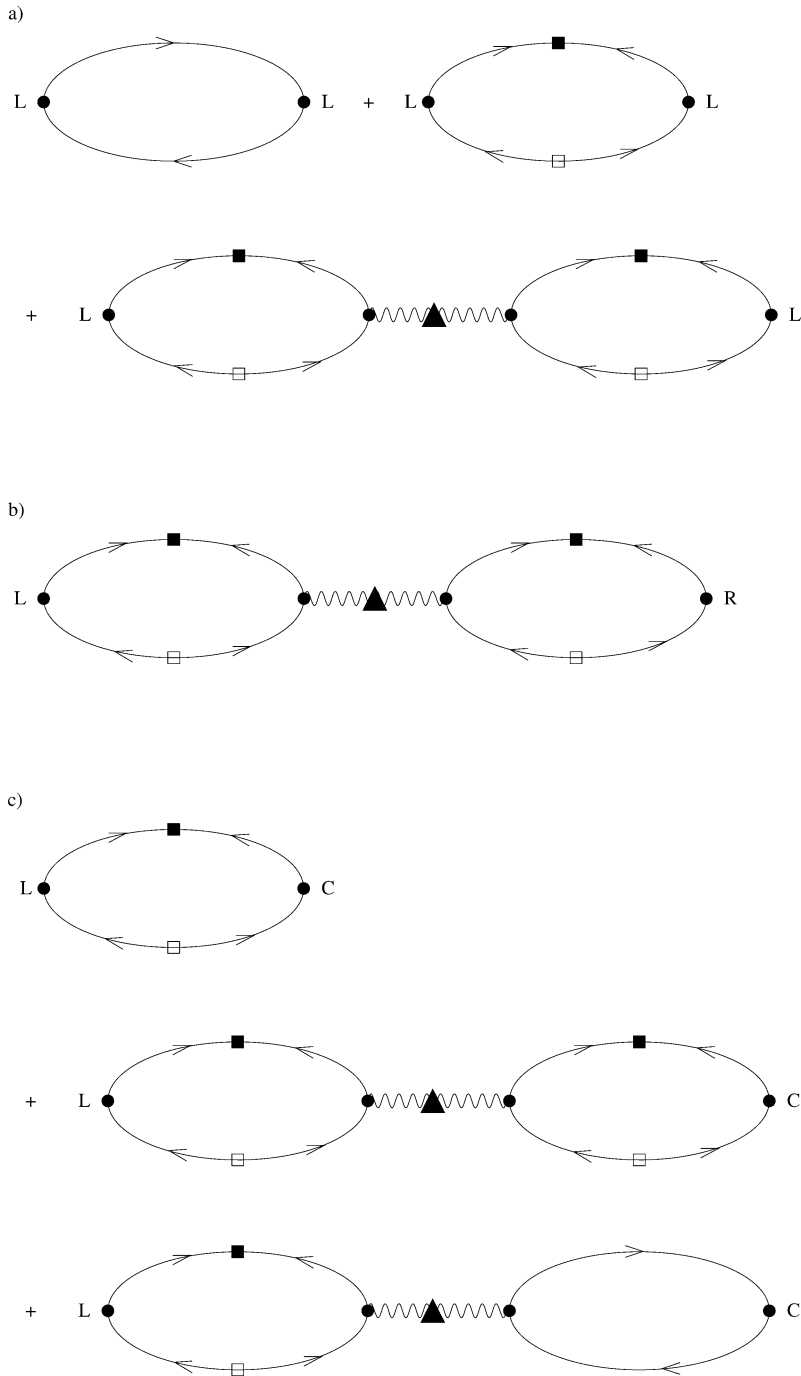


Fig. 3. Diagrams contributing to the two-point functions of two L currents (a), one L and one R current (b), and one L and one color current (c). The squares denote the anomalous fermion self-energy, while the triangle denotes a resummed gluon propagator.

We observe that flavor and color currents are decoupled and that the mixing matrix between left- and right-handed current has the form expected for a system with broken chiral symmetry. To leading order in g^2 , there are no additional contributions to the polarization function in the soft limit. We can now match the result (20) against the low-energy theory:

$$\mathcal{L} = \frac{f_\pi^2}{4} \text{Tr}(\nabla_0 \Sigma \nabla_0 \Sigma^\dagger), \quad (21)$$

where the covariant derivative $\nabla_0 \Sigma = \partial_0 \Sigma + iW_L \Sigma - i\Sigma W_R$ determines the coupling to left- and right-handed gauge fields $W_{L,R}$. Matching the gauge-field mass terms against (20) gives $f_\pi^2 = m_D^2$, which is the result of Son and Stephanov [27,43,44].

This result can also be obtained in a different way. Since the gluon field acquires a large mass of order $g\mu \gg \Delta$, it does not appear in the low-energy effective theory and we should be able to integrate it out [23]. The matrix in (18) has eigenvalues $\lambda = -1/2, -3/2, 0$ and eigenvectors $(1, -1)/\sqrt{2}$, $(1, 1, -2)/\sqrt{6}$ and $(1, 1, 1)/\sqrt{3}$. The vanishing eigenvalue corresponds to the generators of the unbroken $SU(3)_{L+R+C}$. The one-loop polarization function can be matched against the following mass term for the gauge fields:

$$\mathcal{L} = \frac{m_D^2}{4} \left[\frac{1}{2} (W_L - W_R)^2 + \frac{1}{2} (W_L + W_R - 2A_0^T)^2 \right]. \quad (22)$$

The gauge-field mass term still has the structure $\frac{1}{2}(m_D^2/2)(W_L^2 + W_R^2 + \text{mixing})$ apparent in (18). Integrating out the gluon field A_0 eliminates the second term in (22) and we are left with

$$\mathcal{L} = \frac{m_D^2}{4} \frac{1}{2} (W_L - W_R)^2, \quad (23)$$

which has the structure expected from the low-energy effective theory (21). Matching (23) against (21) gives $f_\pi^2 = m_D^2$ as before. The important point is that in both approaches, summing bubble chains or integrating out the gluon field at tree level, the mixing between flavor and color currents cuts down the coefficient of the quadratic terms W_L^2 and W_R^2 by a factor of 2 and introduces mixing between left- and right-handed currents.

We are now in a position to perform the matching calculation for the M^4 term in Eq. (6). In Appendix B we present an alternative argument based on integrating out the gauge field. We are concerned with a possible mass term of the form

$$\begin{aligned} \mathcal{L} &= -\frac{\bar{f}^2}{4} \text{Tr}[(MM^\dagger \Sigma - \Sigma M^\dagger M)(M^\dagger M \Sigma^\dagger - \Sigma^\dagger M M^\dagger)] \\ &= -\frac{\bar{f}^2}{2} \text{Tr}[(MM^\dagger \Sigma M^\dagger M \Sigma^\dagger - M M^\dagger M M^\dagger)]. \end{aligned} \quad (24)$$

We will determine \bar{f} by computing the shift in the ground-state energy proportional to $\text{Tr}[MM^\dagger M^\dagger M]$ and $\text{Tr}[(MM^\dagger)^2]$ in both QCD and in the effective theory. In the effective theory the shift is given by

$$\Delta \mathcal{E} = \frac{\bar{f}^2}{2} \text{Tr}[(M M^\dagger)(M^\dagger M) - (M M^\dagger)^2]. \quad (25)$$

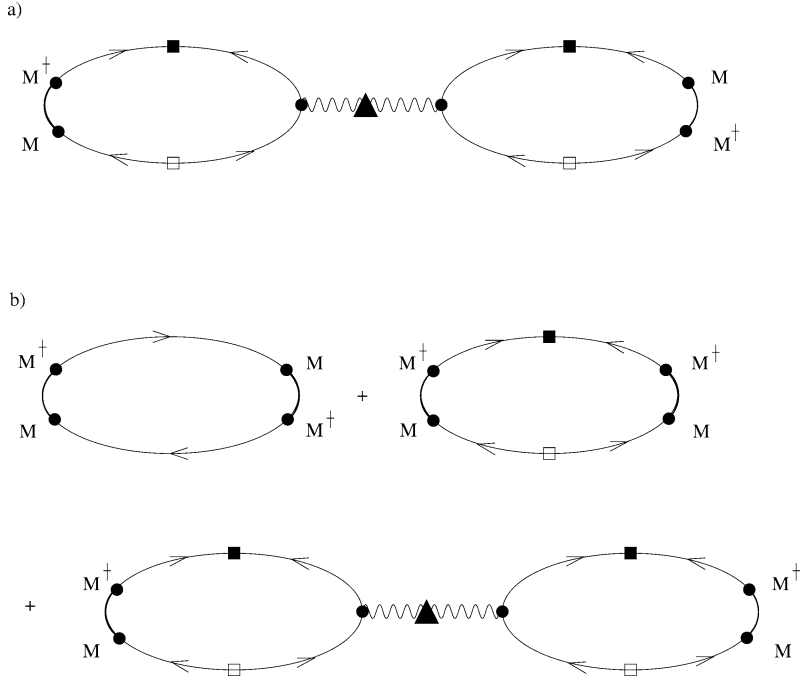


Fig. 4. (a) shows the diagram in the microscopic theory which is matched against the $MM^\dagger \Sigma M^\dagger M \Sigma$ term in the chiral theory. (b) shows the diagrams which are matched against the $(MM^\dagger)^2$ term.

We note that the two terms in Eq. (24) can be distinguished even in the phase $\Sigma = 1$ by the relative position of M and M^\dagger . We also note that other $O(M^4)$ terms allowed by the symmetries of QCD give structures that are different from the ones that appear in Eq. (25).

In the microscopic theory, the shift in the vacuum energy proportional to $\text{Tr}[MM^\dagger M^\dagger M]$ and $\text{Tr}[(MM^\dagger)^2]$ comes from the graphs in Fig. 4, (a) and (b). The $\text{Tr}[MM^\dagger M^\dagger M]$ term is given by

$$\begin{aligned} \Delta \mathcal{E} &= \frac{1}{(2p_F)^2} \left(\frac{m_D^2}{2} \right) \text{Tr}[MM^\dagger \lambda^a] \cdot \frac{\delta^{ab}}{m_D^2} \cdot \left(\frac{m_D^2}{2} \right) \text{Tr}[M^\dagger M \lambda^b] \\ &= \frac{m_D^2}{2} \frac{1}{(2p_F)^2} \text{Tr}[(MM^\dagger)(M^\dagger M)], \end{aligned} \quad (26)$$

and the $\text{Tr}[(MM^\dagger)^2]$ term is

$$\Delta \mathcal{E} = -\frac{m_D^2}{2} \frac{1}{(2p_F)^2} \text{Tr}[(MM^\dagger)^2]. \quad (27)$$

Matching these results against Eq. (25), we conclude that

$$\bar{f}^2 = \frac{f_\pi^2}{(2p_F)^2}, \quad (28)$$

which is the result we derived in Section 2 from making the time derivative covariant with respect to time-dependent flavor transformations.

4. Linear response

In this section we offer a different perspective on the results discussed in the previous sections by using linear response theory. We shall also provide a more microscopic explanation of why the two- and three-flavor cases behave so differently. In the three-flavor case the system responds to a nonzero electron chemical potential by forming a condensate of collective excitations. In the two-flavor case, on the other hand, the response is carried only by the ungapped fermions. From an effective-field theory point of view this is simply due to the fact that three-flavor CFL quark matter has broken chiral symmetry and the low-energy effective description contains charged collective modes whereas the two-flavor theory has unbroken chiral symmetry and the low-energy theory contains ungapped fermions and neutral modes.

In order to set the stage for the discussion of superfluid quark matter we briefly review the response of ordinary quark matter. The grand canonical potential of noninteracting quarks at zero temperature is given by

$$\Omega = -p = -\frac{N_c}{12\pi^2} \sum_f \left[\mu_f k_f \left(\mu_f^2 - \frac{5}{2} m_f^2 \right) + \frac{3}{2} m_f^4 \log \left(\frac{\mu_f + k_f}{m_f} \right) \right], \quad (29)$$

with $k_f = \sqrt{\mu_f^2 - m_f^2}$ is the Fermi momentum and μ_f the chemical potential for the quark flavor $f = u, d, s$. The quark density is given by

$$n_f = -\frac{\partial \Omega}{\partial \mu_f} = \frac{N_c k_f^3}{3\pi^2}. \quad (30)$$

It is convenient to decompose the chemical potential into baryon charge, isospin, and hypercharge components:

$$\mu_u = \mu + \frac{1}{2}\mu_I + \frac{1}{2\sqrt{3}}\mu_Y, \quad \mu_d = \mu - \frac{1}{2}\mu_I + \frac{1}{2\sqrt{3}}\mu_Y, \quad \mu_s = \mu - \frac{1}{\sqrt{3}}\mu_Y.$$

We also note that $\mu_I = \sqrt{3}\mu_Y = -\mu_e$ acts like a chemical potential for electric charge. We can now study the response of the system to an external chemical potential or a change in the quark masses. We begin with the flavor symmetric case $m_u = m_d = m_s = 0$. The isospin and hypercharge susceptibilities are

$$\chi_I = \frac{\partial n_I}{\partial \mu_I} = -\frac{\partial^2 \Omega}{\partial \mu_I^2} = \chi_Y = \frac{\partial n_Y}{\partial \mu_Y} = -\frac{\partial^2 \Omega}{\partial \mu_Y^2} = N_c \left(\frac{\mu^2}{2\pi^2} \right). \quad (31)$$

This result has a very simple interpretation. The change in the isospin or hypercharge density as a function of the corresponding chemical potential is simply given by the density of states on the Fermi surface. The susceptibility (31) can also be calculated in a different way, using the fact that χ is the flavored vector current correlation function at zero momentum. We have

$$\chi_I = -\Pi_I(\omega=0, \vec{k} \rightarrow 0) = -\int d^4x \langle j_0^3(x) j_0^3(0) \rangle \quad (32)$$

with $j_\mu^a(x) = \bar{\psi}(x) \gamma_\mu \frac{\tau^a}{2} \psi$. The correlation function (32) has a vacuum piece and a density-dependent piece. The density-dependent piece is dominated by the contribution of particles

and holes in the vicinity of the Fermi surface. We can calculate this contribution using the effective theory proposed in [24,25]. We get

$$\begin{aligned}\chi_1 &= \lim_{\omega, k \rightarrow 0} N_c \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p_0 - \epsilon_p)(p_0 + \omega - \epsilon_{p+k})} \\ &= N_c \int \frac{d^3 p}{(2\pi)^3} \frac{\partial n}{\partial \epsilon} = N_c \left(\frac{\mu^2}{2\pi^2} \right),\end{aligned}\quad (33)$$

where $\epsilon_p = E_p - \mu$, $E_p = \sqrt{p^2 + m^2}$, and $n(\epsilon)$ is the density of states. This result obviously agrees with Eq. (31).

From the grand canonical potential (29) we can also determine the response of the system to nonzero quark masses. The derivative of the hypercharge density with respect to the strange-quark mass is given by

$$\mu \frac{\partial n_Y}{\partial m_s^2} \Big|_{m_s^2=0} = -\mu \frac{\partial^2 \Omega}{\partial m_s^2 \partial \mu_Y} \Big|_{m_s^2=0} = \frac{N_c}{\sqrt{3}} \left(\frac{\mu^2}{2\pi^2} \right). \quad (34)$$

This result expresses the simple fact that the number of strange quarks is depleted compared to the number of nonstrange quarks as the mass of the strange quark is increased. Again, we can compute this susceptibility using diagrammatic techniques. Computing a one-loop graph with one insertion of μ_Y and one insertion of $m_s^2/(2\mu)$ we reproduce (34).

When we study real physical systems, we are interested in the response of the system subject to the constraint that certain quantities are exactly conserved. In the case of neutron stars, for example, we are interested in the composition of quark matter subject to the condition that the baryon density is fixed and the net density of electric charge is zero. For this purpose we consider the thermodynamic potential as a function of the quark density $\rho_q = 3\rho_B$, the up- and down-quark fractions $x = \rho_u/\rho_q$ and $y = \rho_d/\rho_q$, and the electron chemical potential μ_e :

$$\begin{aligned}\omega(\rho_q, x, y, \mu_e) &= F(\rho_q, x, y) - \mu_e Q \\ &= \frac{3\pi^{2/3}}{4} \rho_q^{4/3} \{x^{4/3} + y^{4/3} + (1-x-y)^{4/3} \\ &\quad + \pi^{-4/3} \rho_q^{-2/3} m_s^2 (1-x-y)^{2/3}\} \\ &\quad + \mu_e \rho_q \left(x - \frac{1}{3} \right) - \frac{1}{12\pi^2} \mu_e^4.\end{aligned}\quad (35)$$

We have neglected higher-order terms in the strange-quark mass as well as the mass of the electron. In order to determine the ground state we have to make (35) stationary with respect to x, y, μ_e . Minimization with respect to x and y enforces β equilibrium, while minimization with respect to μ_e ensures charge neutrality. We find

$$\mu_e \simeq \frac{m_s^2}{4p_F}, \quad (36)$$

which shows that there is a small nonzero μ_e and a corresponding suppression of strange quarks with respect to light quarks even at high density.

We would now like to study how these results are modified in superfluid phases of QCD. We begin with a simple toy model introduced by Rajagopal and Wilczek [17]. The model contains two quark flavors, up and down, that pair in a spin-singlet state which is antisymmetric in both color and flavor. The pair condensate is described by the order parameter $\langle \epsilon^{ab} u^a C \gamma_5 d^b \rangle$. Here, a, b are color indices that only take on the values 1 and 2. One may think of this toy model as $N_f = 2$ QCD where the contribution of the third, unpaired, quark color is ignored. Alternatively, we may think of this theory as $N_c = 2$ QCD.

We can calculate the response in the superfluid in the same way we did in the normal phase, using the relation between the quark-number susceptibilities and the 00-component of the polarization function. In the superfluid phase, there are two contributions coming from the normal and anomalous components of the quark propagator. For the quark number susceptibility we get

$$\begin{aligned} \chi_B &= -\Pi_{00}(\omega=0, \vec{k} \rightarrow 0) \\ &= 4N_c \int \frac{d^4 p}{(2\pi)^4} \left\{ \frac{p_0^2 + \epsilon_p^2}{(p_0^2 - \epsilon_p^2 - \Delta^2)^2} - \frac{\Delta^2}{(p_0^2 - \epsilon_p^2 - \Delta^2)^2} \right\}, \end{aligned} \quad (37)$$

where the first term is the contribution from the normal quark propagator and the second term is the anomalous contribution. The two contributions are exactly equal and sum up to

$$\chi_B = 4N_c \left\{ \left(\frac{\mu^2}{4\pi^2} \right) + \left(\frac{\mu^2}{4\pi^2} \right) \right\} = 4N_c \left(\frac{\mu^2}{2\pi^2} \right), \quad (38)$$

which is equal to the result in the normal phase. We should note that the first term alone only contributes half the susceptibility in the normal phase, even though the susceptibility is independent of the gap and the naive $\Delta \rightarrow 0$ limit of the first graph would seem to correspond to the susceptibility in the normal phase. This is due to the fact that the $\omega \rightarrow 0$ and $\Delta \rightarrow 0$ limits do not commute. This phenomenon is well-known from calculations of the screening mass in other many body systems [45].

The calculation of the isospin susceptibility proceeds along exactly the same lines, only the isospin factors of the two diagrams are different. The isospin factor of the normal contribution is $\text{tr}[\tau_3 \tau_3] = 2$, while the isospin factor of the second term is $\text{tr}[\tau_3 \tau_2 \tau_3 \tau_2] = -2$. The two contributions cancel exactly and the isospin susceptibility is zero. This results has a simple physical interpretation. The superfluid order parameter in $N_f = N_c = 2$ QCD is a flavor singlet and the only broken symmetry is the $U(1)$ of baryon number. As a result, there is only one massless state, the $U(1)$ Goldstone boson. This state couples to the baryon density and leads to a nonzero baryon-number susceptibility but it does not couple to isospin. All states that carry isospin have energies of the order of the gap, so χ_I remains zero as long as $\mu_1 < \Delta$.

We can also see how the calculation of the isospin susceptibility differs in the case of CFL quark matter. Because of the symmetries of the CFL phase, there are two types of quasiparticles, an $SU(3)$ octet with gap $\Delta_8 = \Delta$ and an $SU(3)$ singlet with gap $\Delta_1 = 2\Delta$. Up to degeneracy factors, the two types of quasiparticles contribute equally to the quark-

number susceptibility. We find $\chi_B = 18\mu^2/(2\pi^2)$, which is equal to the result in the normal phase. The calculation of the isospin susceptibility is more complicated. We get

$$\chi_I = 2 \int \frac{d^4 p}{(2\pi)^4} \left\{ \frac{7}{6} \frac{p_0^2 + \epsilon_p^2}{(p_0^2 - \epsilon_p^2 - \Delta_8^2)(p_0^2 - \epsilon_p^2 - \Delta_8^2)} + \frac{1}{3} \frac{p_0^2 + \epsilon_p^2}{(p_0^2 - \epsilon_p^2 - \Delta_8^2)(p_0^2 - \epsilon_p^2 - \Delta_1^2)} - \frac{1}{3} \frac{\Delta_8^2}{(p_0^2 - \epsilon_p^2 - \Delta_8^2)(p_0^2 - \epsilon_p^2 - \Delta_8^2)} - \frac{1}{3} \frac{\Delta_8 \Delta_1}{(p_0^2 - \epsilon_p^2 - \Delta_8^2)(p_0^2 - \epsilon_p^2 - 4\Delta_1^2)} \right\}. \quad (39)$$

The first term comes from particle–hole diagrams with two octet quasiparticles while the second term comes from diagrams with one octet and one singlet quasiparticle. There is no coupling of an octet field to two singlet particles. The third and fourth term are the corresponding contributions from particle–particle and hole–hole pairs. The four integrals in (39) give

$$\chi_I = 2 \left\{ \frac{7}{6} + \frac{1}{3} - \frac{1}{3} - \frac{4 \log(2)}{9} \right\} \left(\frac{\mu^2}{4\pi^2} \right) = \frac{21 - 8 \log(2)}{18} \left(\frac{\mu^2}{2\pi^2} \right) \simeq 0.86 \left(\frac{\mu^2}{2\pi^2} \right), \quad (40)$$

which should be compared to $\chi_I = 3\mu^2/(2\pi^2)$ in the normal phase. We observe that there is a partial cancellation between the normal and anomalous contributions. However, because of the more complicated flavor structure this cancellation is not exact. The isospin density induced by an isospin chemical potential is reduced by a factor ~ 3.5 compared to the normal phase, but it does not vanish. In linear response theory we expand around the point $\mu_I = m_u = m_d = m_s = 0$. In the real world the quark masses are nonzero and there is a critical isospin chemical potential $\mu_I^{\text{crit}} \neq 0$ below which the isospin susceptibility vanishes. In order to see a threshold behavior in μ_I we have to resum mass corrections. This is most efficiently accomplished using the effective chiral description developed in Section 2, see Eq. (10).

5. Summary

We have studied the response of three-flavor quark matter to a nonzero electron chemical potential and a nonzero strange-quark mass. We have focused on the regime $\mu_e, m_s^2/(2p_F) < \Delta$ in which the perturbation does not destroy color–flavor locking. We have identified a new scale $\mu_e, m_s^2/(2p_F) \sim \sqrt{m_{u,d} m_s} (\Delta/p_F)$ which corresponds to the onset of pion or kaon condensation [46–51]. This scale is parametrically much smaller than the gap. If CFL quark matter exists in the core of a neutron star, it is likely to be K^0

condensed. Both with or without a kaon condensate, there are no electrons present [17]. If CFL quark matter is in contact with a hadronic phase that supports a large electron chemical potential the surface layer is likely to be K^- or π^- condensed [52].

These results are based on an analysis of how to incorporate μ_e and $m_s^2/(2p_F)$ in the chiral effective theory. Both terms enter as constant flavor gauge fields, with coefficients completely determined by f_π . The contribution of the $m_s^2/(2p_F)$ term to the Goldstone boson masses is of higher order in the quark masses as compared to the leading-order $\sqrt{mm_s}(\Delta/p_F)$ term. It can nevertheless become dominant because the $O(m)$ term is suppressed by powers of $\sqrt{m/m_s}$ and (Δ/p_F) . As a consequence, the $O(m^2)$ term can cancel the $O(m)$ term without leading to a breakdown of the low-energy expansion.

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Appendix A. Mass terms induced by the color-symmetric diquark condensate

The $O(M^2)$ mass term in (6) gives anomalously small Goldstone-boson masses of the order $m_{GB} \sim \sqrt{mm_s}(\Delta/p_F)$. We already noted that mass terms not suppressed by (Δ/p_F) cannot appear at $O(M^2)$. For strange mesons, the $O(M^2)$ mass term also contains an additional suppression factor $\sqrt{m/m_s}$. Here, m is the mass of the light quarks and m_s is the strange-quark mass. The fact that all Goldstone-boson masses are proportional to the light-quark mass is related to the fact that the CFL order parameter is totally antisymmetric in flavor. This flavor structure also leads to an accidental symmetry of the effective theory at $O(M^2)$. If $m_s = 0$ but $m \neq 0$, we find an octet of exact Goldstone bosons, even though the unbroken flavor symmetry is only $SU(2)$.

There are mass terms at $O(M^2)$ that are consistent with the symmetries of the CFL phase that will remove the accidental symmetry and give contributions to the kaon mass that are proportional to $m_s(\Delta/p_F)$. These terms are induced by the color-flavor symmetric gap parameter

$$\Delta_{ij}^{ab} = \Delta_S (\delta_i^a \delta_j^b + \delta_j^a \delta_i^b). \quad (A.1)$$

The symmetric gap is consistent with the symmetries of the CFL phase but disfavored by the interaction. In particular, one-gluon exchange is repulsive in the color-symmetric quark–quark channel. In perturbative QCD, a small symmetric gap is generated by mixing with the primary gap parameter. We find [6]

$$\Delta_S = \frac{g}{\pi} \frac{\sqrt{2} \log(2)}{36} \Delta_A, \quad (\text{A.2})$$

where Δ_A is the color–flavor antisymmetric gap parameter.

We can calculate the contribution of Δ_S to the Goldstone masses using the methods of Beane et al. [26]. Including the effects of both Δ_A and Δ_S , we find

$$\begin{aligned} \mathcal{L} = & -\frac{\Delta_A \bar{\Delta}_A}{4\pi^2} \log\left(\frac{\Delta_A}{p_F}\right) (\text{Tr}(M \Sigma) \text{Tr}(M \Sigma) - \text{Tr}(M \Sigma M \Sigma) + \text{h.c.}) \\ & -\frac{\Delta_S \bar{\Delta}_S}{4\pi^2} \log\left(\frac{\Delta_S}{p_F}\right) (\text{Tr}(M \Sigma) \text{Tr}(M \Sigma) + \text{Tr}(M \Sigma M \Sigma) + \text{h.c.}). \end{aligned} \quad (\text{A.3})$$

Here, $\bar{\Delta}_{A,S}$ are the flavor-antisymmetric and -symmetric “anti-gaps”. For the purpose of estimating the relative size of the two mass terms, we shall assume that $\bar{\Delta}_{A,S} \simeq \Delta_{A,S}$. We can now calculate the correction to the charged kaon mass:

$$\begin{aligned} m_{K^\pm} &= \left[\frac{4A_A}{f_\pi^2} m_d(m_u + m_s) + \frac{4A_S}{f_\pi^2} (m_u + m_s)(2m_s + 2m_u + m_d) \right]^{1/2} \\ &\simeq \frac{2\sqrt{A_A}}{f_\pi} \sqrt{m m_s} \left(1 + \left(\frac{\Delta_S}{\Delta_A} \right)^2 \frac{m_s}{m_u} + \dots \right), \end{aligned} \quad (\text{A.4})$$

with $A_A = \Delta_A^2/(4\pi^2) \log(p_F/\Delta_A)$. Using (A.2), we observe that the correction term is irrelevant in weak coupling.

Appendix B. $\mu Q + MM^\dagger/(2p_F)$ terms from integrating out the gauge field

Following the discussion in Section 3 we can also derive the $O(M^4)$ terms by integrating out the gauge field. This discussion will also make it clear that the $M^\dagger M$ and MM^\dagger terms enter in the effective Lagrangian like gauge fields, together with flavor-nonsinglet chemical potentials.

In this section we would also like to show how, by explicitly keeping track of the orientation of the CFL order parameter, we can determine how the chiral field Σ enters into the mass terms. This is useful because at higher order the number of independent terms in the chiral Lagrangian quickly proliferates and it becomes more difficult to identify the diagrams in the microscopic theory that correspond to a given term in the effective Lagrangian.

In order to match the microscopic theory against the effective theory in the vacuum ($\Sigma = 1$) phase, we calculate diagrams in the microscopic theory using the Nambu–Gorkov propagators in the normal CFL phase. The inverse Nambu–Gorkov propagator for the ψ_+ field is given by

$$S^{-1} = \begin{pmatrix} p_0 - \epsilon_p & \Delta \\ \Delta & p_0 + \epsilon \end{pmatrix}, \quad (\text{B.1})$$

with the anomalous self-energy

$$(\Delta_L)_{ij}^{ab} = -(\Delta_R)_{ij}^{ab} = \Delta_8 (\delta_i^a \delta_j^b - \delta_j^a \delta_i^b). \quad (\text{B.2})$$

The inverse Nambu–Gorkov propagator is not diagonal in color and flavor. It becomes diagonal in the space spanned by the 9 color–flavor matrices

$$(v^A)^{ai} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\lambda^A)^{ai} & 0 \\ 0 & \frac{1}{\sqrt{2}}(\lambda^A)^{ai} \end{pmatrix}, \quad (\text{B.3})$$

where $\lambda^0 = \sqrt{2/3}$ and λ^A ($A = 1, \dots, 8$) are the Gell-Mann matrices. In this basis, it is straightforward to compute the inverse of (B.1). We find

$$S^{AB} = \frac{\delta^{AB}}{p_0^2 - \epsilon_p^2 - \Delta_A^2} \begin{pmatrix} p_0 + \epsilon & -\Delta^A \\ -\Delta^A & p_0 - \epsilon_p \end{pmatrix}, \quad (\text{B.4})$$

with $\Delta^A = 2\Delta_8$ for $A = 0$ and $\Delta^A = -\text{sym}(A)\Delta_8$ for $A = 1, \dots, 8$. Here, $\text{sym}(A) = 1$ for the symmetric Gell-Mann matrices $A = (1, 3, 4, 6, 8)$ and $\text{sym}(A) = -1$ for the antisymmetric matrices $A = (2, 5, 7)$.

In order to keep the dependence on Σ , we have to perform the calculation using the anomalous self-energy in the rotated vacuum:

$$(\Delta_L)_{ij}^{ab} = \Delta_8(X_i^a X_j^b - X_j^a X_i^b), \quad -(\Delta_R)_{ij}^{ab} = \Delta_8(Y_i^a Y_j^b - Y_j^a Y_i^b), \quad (\text{B.5})$$

with $X \in SU(3)_L$ and $Y \in SU(3)_R$. The Nambu–Gorkov propagator for left-handed fermions is diagonal in a basis spanned by the color–flavor matrices

$$(\tilde{v}_L^A)^{ai} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\lambda^A X^T)^{ai} & 0 \\ 0 & \frac{1}{\sqrt{2}}(\lambda^A X^T)^{ai} \end{pmatrix}, \quad (\text{B.6})$$

with a similar set of matrices $(\tilde{v}_R^A)^{ai}$ which diagonalize the propagator for right-handed fermions. In the basis (B.6), the fermion propagator in the rotated CFL vacuum has exactly the same form (B.4) that it had in the ordinary CFL vacuum (B.2). The dependence on X, Y comes in when we calculate diagrams with external color or flavor currents. In that case we have to take matrix elements of the external current between the basis states (\tilde{v}_L) and (\tilde{v}_R) .

We can now calculate a one-loop diagram with insertions of MM^\dagger and the gauge field A_0 . We find

$$\Delta\mathcal{E} = \frac{m_D^2}{2p_F} \text{Tr}[X^\dagger M M^\dagger X A_0^T]. \quad (\text{B.7})$$

In the same way, we also calculate diagrams with insertions of $M^\dagger M$ and Q . Collecting all these terms, we get

$$\mathcal{E} = \frac{m_D^2}{2} \text{Tr} \left[\left(X^\dagger \mu_e Q X + X^\dagger \frac{M M^\dagger}{2p_F} X + A_0^T \right)^2 + \left(Y^\dagger \mu_e Q^\dagger Y + Y^\dagger \frac{M^\dagger M}{2p_F} Y + A_0^T \right)^2 \right]. \quad (\text{B.8})$$

Similar to the calculation of f_π it is essential here to take into account the mixing with the gauge field. Without the A_0 field, we would conclude that there is no dependence on the flavor matrices X, Y . We can now integrate out the gauge field A_0 . We get

$$\Delta\mathcal{E} = \frac{m_D^2}{4} \text{Tr} \left[\left(\left(\mu_e Q + \frac{MM^\dagger}{2p_F} \right) \Sigma - \Sigma \left(\mu_e Q^\dagger + \frac{M^\dagger M}{2p_F} \right) \right) \right. \\ \left. \times \left(\left(\mu_e Q^\dagger + \frac{M^\dagger M}{2p_F} \right) \Sigma^\dagger - \Sigma^\dagger \left(\mu_e Q + \frac{MM^\dagger}{2p_F} \right) \right) \right], \quad (\text{B.9})$$

where $\Sigma = XY^\dagger$. We note that, after integrating out the gauge field, the vacuum energy (B.9) only depends on the chiral field Σ and not on X and Y separately. Using $f_\pi = m_D$, we observe that (B.9) contains the terms required to complete the covariant derivative in (6).

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